1. (12 points) On the line provided for each part, list all possible values which answer the question.
If there is no possible value, write **DNE**. No justifications are needed for this question

Suppose A is a 5×9 matrix and the associated linear transformation L_A, where $L_A(\mathbf{x}) = A\mathbf{x}$, is surjective.

Suppose B is a 7×4 matrix and the associated linear transformation L_B, where $L_B(\mathbf{x}) = B\mathbf{x}$, is injective.

(a) What are all of the possible values for the nullity of A?

(a) **4 only**

(b) What are all of the possible values for the rank of A?

(b) **5 only**

(c) What are all of the possible values for the rank of A^T?

(c) **5 only**

(d) What are all of the possible values for the rank of B?

(d) **4 only**

(e) What are all of the possible values for the nullity of B?

(e) **0 only**

(f) What are all of the possible values for the nullity of B^T?

(f) **3 only**

2. (12 points) On the line for each part, provide one matrix that satisfies the condition.
If no such matrix exists, write **DNE**. No justifications are needed for this question.

(a) 3×3 matrix A where $\dim(\text{Col}(A)) = 1$.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(b) 3×3 matrix B where $\dim(\text{Row}(B)) = 2$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) 3×3 matrix C where $\dim(\text{Nul}(C)) = 3$.

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) 2×4 matrix D where $\dim(\text{Nul}(D)) = 3$.

(d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(e) 2×4 matrix E where $\dim(\text{Row}(E)) = 3$.

(e) **DNE**

(f) 2×4 matrix F where $\dim(\text{Col}(F)) = 3$.

(f) **DNE**

3. (10 points) Let $S = \left\{ \begin{bmatrix} 4 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix} \right\}$. Extend S to a basis for $\mathbb{R}_{2\times 2}$.

> These two independent vectors in $\mathbb{R}_{2\times 2}$ have coordinate vectors $\begin{bmatrix} 4 \\ 3 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 2 \\ 6 \end{bmatrix} \in \mathbb{R}^4$. At least two of the standard basis vectors for $\mathbb{R}^4$ could be added to this set to form a basis of $\mathbb{R}^4$. Reduce $\left[\begin{array}{cc|cccc} 4 & 4 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$ to an echelon form to see that any two can be chosen. A possible answer could be $\left\{ \begin{bmatrix} 4 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, using $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^4$.

4. (10 points) Let $T = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \right\}$. Reduce T to a basis for $\mathbb{R}_{2\times 2}$.

> The last vector is a scalar multiple of the fourth, so it can be removed immediately. Using coordinate vectors for the remaining five, reduce $\begin{bmatrix} 2 & 1 & 0 & 4 & 3 \\ 1 & 1 & 0 & 4 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 8 & 6 \end{bmatrix}$ to an echelon form to see that the first four vectors form a basis of $\mathbb{R}^4$. A possible answer could be $\left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix} \right\}$.

5. (12 points) Let $\mathcal{E}$ be the standard basis for $\mathbb{R}^3$ and let $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ be another basis for $\mathbb{R}^3$.

(a) Find $[\mathbf{x}]_{\mathcal{E}}$, the coordinate vector of $\mathbf{x} = \begin{bmatrix} 4 \\ 7 \\ -8 \end{bmatrix}$ with respect to $\mathcal{E}$. Justify your answer.

$[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 7 \\ -8 \end{bmatrix}$ because $\mathbf{x} = 4\mathbf{e}_1 + 7\mathbf{e}_2 - 8\mathbf{e}_3$.

(b) Find $[\mathbf{y}]_{\mathcal{B}}$, the coordinate vector of $\mathbf{y} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$ with respect to $\mathcal{B}$. Justify your answer.

$[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 2 \end{bmatrix}$ because $\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

(c) Is there a vector $\mathbf{z}$ such that $[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}$? Justify your answer.

Yes. $\mathbf{z} = 7 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 28 \\ -21 \\ -2 \end{bmatrix}$.

6. (12 points) Let $L : V \to W$ be a linear transformation where $\dim(V) = 5$ and $\dim(W) = 4$.
Determine the validity of each statement as **T** (always) TRUE, **F** (always) FALSE, or **S** SOMETIMES true/false. Provide a brief justification for your answer.

(a) L is an isomorphism. (An isomorphism is a bijective linear transformation.)

(a) **F**

The matrix for L is 4×5 which has a maximum rank of 4, even though it has 5 columns. L cannot be injective.

(b) $\dim(\ker(L)) = 5$.

(b) **S**

Example when true: The trivial map $T : \mathbb{R}^5 \to \mathbb{R}^4$ via $T(\mathbf{x}) = \mathbf{0}$ has a 5-dimensional kernel.
Example when false: The derivative map $\frac{d}{dt} : \mathbb{P}_4 \to \mathbb{P}_3$ via $\mathbf{p}(t) \mapsto \mathbf{p}'(t)$ has a 1-dimensional kernel.

(c) If H is a subspace of V then $\dim(H) < \dim(L(H))$.

(c) **F**

A basis for H cannot have less elements than a basis for $L(H)$. If L is injective then the dimensions could be equal, at best.

7. (8 points) Let A, B, C, D be 3×3 matrices such that $\det(A) = \frac{1}{4}$, $\det(B) = \sqrt{3}$, $\det(C) = 2$, $\det(D) = 0$.

(a) Compute $\det(AD) - \det(BC)$.

$0 - 2\sqrt{3}$

(b) Compute $\det(B^2 A^T) + \det\left(2C^{-1}\right)$.

$\left(\sqrt{3}\right)^2 \left(\frac{1}{4}\right) + 2^3 \left(\frac{1}{2}\right) = \frac{3}{4} + 4 = \frac{19}{4}$

8. (4 points) Use determinants to determine if the the three points $(-3, -8)$, $(1, -3)$, $(5, 2) \in \mathbb{R}^2$ are collinear.
(Hint: Otherwise they would form a triangle.)

Translate all three points so that one of them is at the origin, then use the other two as the terminal points of two vectors which generate the “triangle.” Compute $\begin{vmatrix} -4 & 4 \\ -5 & 5 \end{vmatrix} = 0$. Since the area of this “triangle” is 0, the points must be collinear.

Alternatively, we can compute $\begin{vmatrix} -3 & -8 & 1 \\ 1 & -3 & 1 \\ 5 & 2 & 1 \end{vmatrix} = 0$ to determine that the points are collinear.

9. (10 points) Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 4 & -2 & 2 \\ -7 & 1 & -6 \end{bmatrix}$.

(a) Determine if $\mathbf{e}_2 \in \text{Nul}(A)$.

$\mathbf{e}_2 \notin \text{Nul}(A)$. Compute $\begin{bmatrix} 1 & -2 & -1 \\ 4 & -2 & 2 \\ -7 & 1 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) Determine if $\mathbf{e}_3 - \mathbf{e}_1 \in \text{Col}(A)$.

$\mathbf{e}_3 - \mathbf{e}_1 \notin \text{Col}(A)$. The augmented column of $\left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 4 & -2 & 2 & 0 \\ -7 & 1 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ has a pivot. Therefore $\mathbf{e}_3 - \mathbf{e}_1$ is not a linear combination of the columns of A.

(c) Compute $\dim(\text{Row}(A)) - \dim(\text{Row}(A^T))$.

A and A^T have the same rank, so $\dim(\text{Row}(A)) - \dim(\text{Row}(A^T)) = 0$.

10. (10 points) Let $A \in \mathbb{R}_{m\times p}$ and $B \in \mathbb{R}_{n\times p}$ and let $U = \{\mathbf{x} \in \mathbb{R}^p \mid A\mathbf{x} = B\mathbf{x}\}$.

(a) When $m \neq n$, U is a not a subspace of $\mathbb{R}^p$. Why not?

When $m \neq n$, $A\mathbf{x} \in \mathbb{R}^m$ but $B\mathbf{x} \in \mathbb{R}^n$. Therefore U is an empty set, and hence not a subspace of $\mathbb{R}^p$.

(b) When $m = n$, decide if U can be a subspace of $\mathbb{R}^p$. Fully justify your answer.

When $m = n$, U can be described as $\{\mathbf{x} \in \mathbb{R}^p \mid A\mathbf{x} - B\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^p \mid (A - B)\mathbf{x} = \mathbf{0}\} = \text{Nul}(A - B)$. Thus U is a subspace of $\mathbb{R}^p$ simply because it is a *Fundamental Subspace* of $A - B$.

Alternatively, we can just go through the subspace test.

- U is nonempty since $A\mathbf{0} = B\mathbf{0} = \mathbf{0}$.
- U is closed under vector addition since for any $\mathbf{x}, \mathbf{y} \in U$, $A\mathbf{x} = B\mathbf{x}$ and $A\mathbf{y} = B\mathbf{y}$.
  $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = B\mathbf{x} + B\mathbf{y} = B(\mathbf{x} + \mathbf{y})$.
- U is closed under scalar multiplication since for any $\mathbf{x}, \in U$ and any $k \in R$,
  $A(k\mathbf{x}) = kA\mathbf{x} = kB\mathbf{x} = B(kA\mathbf{x})$.